Apr. 2010

CHINESE JOURNAL
OF ENGINEERING MATHEMATICS

Vol. 27 No. 2

Article ID:1005-3085(2010)02-0369-06

Characteristic-mixed Methods on Dynamic Finite Element Spaces for Two-phase Miscible Flow in Porous Media*

FU Hong-fei[1], RUI Hong-xing[2]

(1- School of Mathematics and Computational Science, China University of Petroleum, Dongying 257061; 2- School of Mathematics, Shandong University, Jinan 250100)

Abstract: The miscible displacement of one incompressible fluid by another in a porous media is governed by a nonlinear coupled system of two partial differential equations, one of elliptic form for the pressure and the other of parabolic form for the concentration of the fluids. In this paper, the concentration is approximated by a modified method of characteristics (MMOC) combined with dynamic finite element spaces, while the pressure and Darcy velocity of the mixture are approximated simultaneously by a mixed finite element method. By adopting the negative-norm estimate, convergence analysis and error estimates are established.

Keywords: dynamic finite element spaces; MMOC; mixed finite element method; negative-norm

Classification: AMS(2000) 65M15; 65M60 **CLC number:** O241.82 **Document code:** A

1 Introduction

Miscible displacement is an enhanced oil-recovery process that has attracted considerable attention in the petroleum industry over last 60 years. It involves injection of a solvent at certain wells in a petroleum reservoir, with the intention of displacing resident oil to other wells for production. This oil may have been left behind after primary production by reservoir pressure or secondary production by waterflooding. The economics of the process can be precarious, because the chemicals it requires are expensive and the performance of the displacement is by no means guaranteed. Mathematically, the process, i.e., miscible displacement of one incompressible fluid by another in a porous media $\Omega \subset \mathbf{R}^2$ over time interval $J = (0, T]$ is modeled by a nonlinear coupled system of two partial differential equations.

In this paper, a dynamic finite element method combined with the MMOC is applied to the concentration equation, while the pressure equation is approximated by a mixed finite element method. The main idea of dynamic finite element method is that a different number of finite element spaces is adopted at different time level, and the approximate solution at the current time is projected in the L^2-norm onto the next time finite element space and make it as an initial value. For the theoretical analysis and practical computation of dynamic finite element

Received: 23 May 2007. **Biography**: Fu Hongfei (Born in 1981), Male, Doctor.

Accepted: 12 Nov 2008. Research field: numerical analysis for partial differential equation.

***Foundation item:** The National Natural Science Foundation of China (10771124).

methods, we refer the reader to [1-5] and the references cited therein. By adopting the negative-norm estimate, we prove that when $Mh^2/\Delta t$ is bounded, the L^2-norm is optimal, and under the condition that $Mh^4/\Delta t$ is bounded, optimal error estimate in energy norm is showed.

2 Characteristic-mixed methods with dynamic finite element spaces

The mathematical model of miscible displacement problem in petroleum reservoir simulation is of the following form

$$\begin{cases} (a) \quad -\nabla\cdot(a(c)(\nabla p-\gamma(c)\nabla d))\equiv\nabla\cdot u=q(x,t), & x\in\Omega, \quad t\in J,\\ (b) \quad \phi\frac{\partial c}{\partial t}+u\cdot\nabla c-\nabla\cdot(D(x,u)\nabla c)=(\tilde{c}-c)\tilde{q}, & x\in\Omega, \quad t\in J,\\ (c) \quad u\cdot n=D(u)\nabla c\cdot n=0, & x\in\partial\Omega, \quad t\in J,\\ (d) \quad c(x,0)=c_0(x), & x\in\Omega, \quad t=0. \end{cases} \tag{1}$$

For detailed meanings of the parameters and some regular hypotheses of the solutions and coefficients in problem (1), we refer to [6-8], for example.

Basically, the methods of characteristics are to think of the hyperbolic part of Eq (1b), namely $\phi\partial c/\partial t+u\cdot\nabla c$, as a directional derivative. Accordingly, let s denote the unit vector in the direction of (u_1,u_2,ϕ) in $\Omega\times J$, and set

$$\psi(x)=\left(|u(x)|^2+\phi(x)^2\right)^{1/2}=\left(u_1(x)^2+u_2(x)^2+\phi(x)^2\right)^{1/2}.$$

Then Eq (1b) can be written in the following form

$$\psi\frac{\partial c}{\partial s}-\nabla\cdot(D\nabla c)+\tilde{q}c=\tilde{q}\tilde{c}. \tag{2}$$

Let $V=H(\mathrm{div};\Omega)\cap\{v\cdot n=0 \text{ on } \partial\Omega\}$, $W=\{q\in L^2(\Omega):\int_\Omega q\,\mathrm{d}x=0\}$. For $\alpha,\ \beta\in V,\ \phi\in W$, and $\theta\in L^\infty(\Omega)$ define the following bilinear forms

$$A(\theta;\alpha,\beta)=\left(\frac{1}{a(\theta)}\alpha,\beta\right),\quad B(\alpha,\phi)=-(\nabla\cdot\alpha,\phi).$$

Then we obtain the following equivalent weak formula of problem (1)

$$\begin{cases} (a) \quad (\psi\frac{\partial c}{\partial s},z)+(D(u)\nabla c,\nabla z)+(\tilde{q}c,z)=(\tilde{q}\tilde{c},z), & z\in H^1(\Omega),\\ (b) \quad A(c;u,v)+B(v,p)=(\gamma(c)\nabla d,v), & v\in V,\\ (c) \quad B(u,\phi)=-(q,\phi), & \phi\in W, \end{cases} \tag{3}$$

where $0<t\le T$ and $c(x,0)=c_0(x)$.

Suppose the time step $\Delta t=T/N$ with $t_n=n\Delta t,\ n=0,1,\cdots,N$, where N is a positive integer. Let $\widetilde{V}_n\times\widetilde{W}_n$ be a Raviart-Thomas space of index k_n associating with a quasi-regular triangulation of Ω, such that the elements have diameters bounded by h_p^n in every time t_n. Define $V_n=\{v\in\widetilde{V}_n;v\cdot n=0 \text{ on } \partial\Omega\}$, $W_n=\widetilde{W}_n/\{\phi\equiv\text{constant on }\Omega\}$. It is clear that $V_n\times W_n\subset V\times W$. Next let $M_n\subset W_\infty^1(\Omega)$ be a standard finite element space for a Galerkin

method of index l_n associated with another quasi-regular polygonalization of Ω, such that the elements have diameters bounded by h_c^n. Set

$$k = \inf_{0\le n\le N} k_n, \quad l = \inf_{0\le n\le N} l_n, \quad h_c = \sup_{0\le n\le N} h_c^n, \quad h_p = \sup_{0\le n\le N} h_p^n.$$

The reader should not be confused with $h_c^n(h_p^n)$ and $h_c^k(h_p^k)$, where the former is the spatial step size at time t_n, and the latter means $h_c(h_p)$ to the kth power.

At each time level n, we assume $\{c^n, u^n, p^n\}$ is approximated by $\{C^n, U^n, P^n\} \in M_n \times V_n \times W_n$, respectively, and for the initial approximation, we assume C^0 is the elliptic projection of $c_0(x)$ onto M_0. Now we define our approximation scheme as follows.

1) Find $\hat{C}^n \in M_{n+1}$, such that

$$(\phi \hat{C}^n, z) = (\phi C^n, z), \quad z \in M_{n+1}, \quad n = 0, 1, \cdots, N-1. \tag{4}$$

2) When $\hat{C}^n$ is known, we get $\{C^{n+1}, U^{n+1}, P^{n+1}\} \in M_{n+1} \times V_{n+1} \times W_{n+1}$, such that for $z \in M_{n+1}, v \in V_{n+1}$ and $w \in W_{n+1}$

$$\Big(\phi \frac{C^{n+1} - \check{\hat{C}}^n}{\Delta t_{n+1}}, z\Big) + \big(D(U^{n+1})\nabla C^{n+1}, \nabla z\big) + (\tilde{q}^{n+1} C^{n+1}, z) = (\tilde{q}^{n+1}\tilde{c}^{n+1}, z), \tag{5}$$

$$A(\hat{C}^n; U^{n+1}, v) + B(v, P^{n+1}) = \big(\gamma(\hat{C}^n)\nabla d, v\big), \tag{6}$$

$$B(U^{n+1}, w) = -(q^{n+1}, w), \quad n = 0, 1, \cdots, N-1, \tag{7}$$

where

$$\check{\hat{C}}^n = \hat{C}^n(\check{x}) = \hat{C}^n\Big(x - \frac{U^{n+1}(x)}{\phi(x)}\Delta t\Big).$$

Noting that from (4), we can easily see that when different finite element mesh or interpolation functions are used at time level $t = t_n$ and $t = t_{n+1}$, we must project the approximate solution C^n to the next time finite element space M_{n+1}, and make it as an initial value. In practice, once we know $\hat{C}^n$ by (4), we can obtain $\{U^{n+1}, P^{n+1}\}$ by (6)-(7), and then get C^{n+1} by (5).

For convenience of theoretical analysis in the following section, we introduce the following two useful projections

$$\begin{cases} A(c(t); R_n u(t), v) + B(v, R_n p(t)) = (\gamma(c(t))\nabla d, v), & v \in V_n, \quad t \in J, \\ B(R_n u(t), w) = -(q(t), w), & w \in W_n, \quad t \in J. \end{cases} \tag{8}$$

$$\begin{aligned} &(D(u(t))\nabla(R_n c(t)), \nabla z) + (R_n c(t), z) + (\tilde{q}(t) R_n c(t), z) \\ &= (D(u(t))\nabla c(t), \nabla z) + (c(t), z) + (\tilde{q}(t) c(t), z) \\ &= -\Big(\phi \frac{\partial c}{\partial t}(t), z\Big) - (u(t)\cdot \nabla c(t), z) + (c(t), z) + (\tilde{q}(t)\tilde{c}(t), z), \quad z \in M_n, \quad t \in J. \end{aligned} \tag{9}$$

For the projection solutions defined in (8)-(9), we have the following approximate results.

Lemma 1[1] There exist a constant K independent of h_p and h_c, such that

$$\|u - R_n u\|_{L^\infty(J;H(\mathrm{div}))} + \|p - R_n p\|_{L^\infty(J;L^2)} \le K\big(\|u\|_{L^\infty(J;H^{k+1}(\mathrm{div}))} + \|p\|_{L^\infty(J;H^{k+1})}\big)h_p^{k+1},$$

$$\|c - R_n c\|_{L^\infty(J;L^2)} + h_c\|c - R_n c\|_{L^\infty(J;H^1)} \le K\|c\|_{L^\infty(J;H^{l+1})} h_c^{l+1}.$$

3 Main results

In this section, we shall show that the optimal-order convergence rate rests with the total times of changing finite element spaces and discrete parameters. Set

$$e^n = U^n - R_n u^n,\ \theta^n = C^n - R_n c^n,\ \rho^n = u^n - R_n u^n,\ \lambda^n = c^n - R_n c^n,\ n = 0, 1, \cdots, N,$$

$$\hat{\lambda}^n = c^n - R_{n+1} c^n,\ \hat{\theta}^n = \hat{C}^n - R_{n+1} c^n,\ n = 0, 1, \cdots, N-1.$$

Lemma 2 Suppose that $\{C, U, P\}$ are the solutions of (4)-(7) and $\{Rc, Ru, Rp\}$ are defined by (8)-(9), respectively. Then

$$\|U^{n+1} - R_{n+1} u^{n+1}\|_V + \|P^{n+1} - R_{n+1} p_{n+1}\|_W \leq K\|c^{n+1} - \hat{C}^n\|. \tag{10}$$

Lemma 3 Suppose that $\{C, U, P\}$ are the solutions of (4)-(7) and $\{Rc, Ru, Rp\}$ are defined by (8)-(9), respectively. Then

$$\begin{aligned} &\|\phi^{\frac{1}{2}}\theta^{n+1}\|^2 - \|\phi^{\frac{1}{2}}\theta^n\|^2 + \Delta t(D(U^{n+1})\nabla\theta^{n+1}, \nabla\theta^{n+1}) \\ \leq\ & K\{\Delta t(\|\theta^{n+1}\|^2 + \|\hat{\theta}^n\|^2) + E_n\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} E_n =\ & \Delta t^2\Big(\Big\|\frac{\partial^2 c}{\partial \tau^2}\Big\|^2_{L^2(t_n,t_{n+1};L^2)} + \Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(t_n;t_{n+1};L^2)}\Big) + h_c^{2l+2}\Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(t_n;t_{n+1};H^{l+1})} \\ & +(\Delta t)^{-1} h_c^{2l+4}\|c\|^2_{L^\infty(J;H^{l+1})} + \Delta t\ h_c^{2l+2}\|c\|^2_{L^\infty(J;H^{l+1}(\Omega))} \\ & +\Delta t\ h_p^{2k+2}(\|u\|^2_{L^\infty(J;H^{k+1}(div))} + \|p\|^2_{L^\infty(J;H^{k+1}(\Omega))}). \end{aligned}$$

Proof It follows from (5) and (9) as well as the L^2-projection (4) that

$$\begin{aligned} & \Big(\phi\frac{\theta^{n+1} - \theta^n}{\Delta t}, z\Big) + (D(U^{n+1})\nabla\theta^{n+1}, \nabla z) \\ =\ & \Big(\Big[\phi\frac{\partial c^{n+1}}{\partial t} + U^{n+1}\cdot\nabla c^{n+1}\Big] - \phi\frac{c^{n+1} - \check{c}^n}{\Delta t}, z\Big) - (\lambda^{n+1}, z) - (\tilde{q}^{n+1}\theta^{n+1}, z) \\ & +([u^{n+1} - U^{n+1}]\cdot\nabla c^{n+1}, z) + ([D(u^{n+1}) - D(U^{n+1})]\cdot\nabla R_{n+1}c^{n+1}, \nabla z) \\ & +\Big(\phi\frac{\lambda^{n+1} - \hat{\lambda}^n}{\Delta t}, z\Big) + \Big(\phi\frac{\hat{\lambda}^n - \lambda^n}{\Delta t}, z\Big) + \Big(\phi\frac{\hat{\lambda}^n - \check{\hat{\lambda}}^n}{\Delta t}, z\Big) - \Big(\phi\frac{\hat{\theta}^n - \check{\hat{\theta}}^n}{\Delta t}, z\Big). \end{aligned} \tag{12}$$

Let $z = \theta^{n+1}$ be the test function, for the estimates of the right-hand side of (12), we should only pay special attention to the estimate of the seventh term. Since if $M_n = M_{n+1}$, i.e., the same finite element spaces between the time levels t_n and t_{n+1} are used, then it follows from (4) that $\lambda^n = \hat{\lambda}^n$, and so that this term reduces to zero. Otherwise, by utilizing the negative-norm estimate, we have

$$\begin{aligned} \Big|\Big(\phi\frac{\hat{\lambda}^n - \lambda^n}{\Delta t}, \theta^{n+1}\Big)\Big| &\leq K\Big\|\frac{\hat{\lambda}^n - \lambda^n}{\Delta t}\Big\|_{-1}\|\theta^{n+1}\|_1 \\ &\leq K(\Delta t)^{-2}h_c^{2l+4}\|c\|^2_{L^\infty(J;H^{l+1})} + \epsilon\|\theta^{n+1}\|_1^2. \end{aligned} \tag{13}$$

Then we conclude by Lemmas 1, 2, and the well known estimates for MMOC that

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big[\|\phi^{\frac12}\theta^{n+1}\|^2-\|\phi^{\frac12}\theta^{n}\|^2\Big]+\big(D(U^{n+1})\nabla\theta^{n+1},\nabla\theta^{n+1}\big)\\
\le\ & K\|\theta^{n+1}\|^2+K\|\hat\theta^{n}\|^2+K\Delta t\Big(\Big\|\frac{\partial^2 c}{\partial\tau^2}\Big\|^2_{L^2(t_n,t_{n+1};L^2)}+\Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(t_n;t_{n+1};L^2)}\Big)\\
&+K(\Delta t)^{-1}h_c^{2l+2}\Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(t_n;t_{n+1};H^{l+1})}+K(\Delta t)^{-2}h_c^{2l+4}\|c\|^2_{L^\infty(J;H^{l+1})}\\
&+Kh_c^{2l+2}\|c\|^2_{L^\infty(J;H^{l+1}(\Omega))}+Kh_p^{2k+2}\big(\|u\|^2_{L^\infty(J;H^{k+1}(div))}+\|p\|^2_{L^\infty(J;H^{k+1}(\Omega))}\big)\\
&+\epsilon\|\nabla\theta^{n+1}\|^2. \qquad (14)
\end{aligned}$$

Multiplying (14) by $2\Delta t$ and choosing ϵ sufficiently small, we then complete the proof.

Lemma 4 *Let M be the total times of changing finite element spaces for concentration in the time direction, then we have*

$$\begin{aligned}
&\|\theta^N\|^2+\sum_{n=0}^{N-1}\Delta t\|\nabla\theta^{n+1}\|^2\\
\le\ & K\Delta t^2\Big(\Big\|\frac{\partial^2 c}{\partial\tau^2}\Big\|^2_{L^2(J;L^2)}+\Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(J;L^2)}\Big)+Kh_c^{2l+2}\Big\|\frac{\partial c}{\partial t}\Big\|^2_{L^2(J;H^{l+1})}\\
&+Kh_p^{2k+2}\big(\|u\|^2_{L^\infty(J;H^{k+1}(div))}+\|p\|^2_{L^\infty(J;H^{k+1}(\Omega))}\big)\\
&+K\Big(\frac{Mh_c^2}{\Delta t}+1\Big)h_c^{2l+2}\|c\|^2_{L^\infty(J;H^{l+1}(\Omega))}. \qquad (15)
\end{aligned}$$

Proof *It follows immediately from Lemma 3 and the projective relation (4).*

Combing the estimate in Lemma 4 with that in Lemma 1, it is easy to get the following main result.

Theorem 1 Suppose $\{c,u,p\}$ and $\{C,U,P\}$ are the solutions of (1) and (4)-(7), respectively. Under some regularity assumptions on the solutions and coefficients, we have the following error estimates for the concentration that

$$\max_{0\le n\le N}\|c^n-C^n\|^2=\mathcal{O}\Big(\Delta t^2+\Big(\frac{Mh_c^2}{\Delta t}+1\Big)h_c^{2l+2}+h_p^{2k+2}\Big), \qquad (16)$$

$$\sum_{n=1}^{N}\Delta t\|\nabla(c^n-C^n)\|^2=\mathcal{O}\Big(\Delta t^2+\Big(\frac{Mh_c^4}{\Delta t}+1\Big)h_c^{2l}+h_p^{2k+2}\Big). \qquad (17)$$

The size of Δt term depends principally on $\|\frac{\partial^2 c}{\partial\tau^2}\|$ and $\|\frac{\partial c}{\partial t}\|$. The spatial terms depends principally on the H^{l+1} and H^{k+1} norms of c and u, p.

Theorem 2 Under the assumptions of Theorem 1 and the extra assumption that $c\in W^1_\infty(J;L^2)$, the errors in velocity and pressure are bounded by

$$\max_{1\le n\le N}\big(\|u^n-U^n\|^2_V+\|p^n-P^n\|^2_W\big)=\mathcal{O}\Big(\Delta t^2+\Big(\frac{Mh_c^2}{\Delta t}+1\Big)h_c^{2l+2}+h_p^{2k+2}\Big). \qquad (18)$$

References:

[1] Miller K, Miller R. Moving finite elements[J]. SIAM Journal on Numerical Analysis, 1981, 18: 1019-1057
[2] Liang G P. A FEM with moving grid[J]. Mathematica Numerica Sinica, 1985, 4: 377-384
[3] Yang D Q. The mixed finite element methods with moving grids for parabolic problems[J]. Mathematica Numerica Sinica, 1988, 10: 266-271
[4] Yuan Y R. On characteristic finite element methods with moving mesh for nonlinear convection diffusion problems[J]. Numerical Mathematics: A Journal of Chinese University, 1983, 8: 236-245
[5] Yang D Q. A characteristic mixed method with dynamic finite-element space for convection-dominated diffusion problems[J]. Journal of Computational and Applied Mathematics, 1992, 43: 343-353
[6] Douglas J, Ewing R E, Wheeler M F. The approximation of the pressure by a mixed method in the simulation of miscible displacement[J]. RAIRO Analysis Numéric, 1983, 17: 17-33
[7] Ewing R E, Russell T F, Wheeler M F. Convergence analysis of an approximation of miscible displacement in porous media by mixed finite element and a modified method of characteristics[J]. Computer Methods in Applied Mechanics and Engineering, 1984, 47: 73-92
[8] Song H L. Analysis of an approximation of miscible displacement in porous media by Godunov-mixed methods[J]. Chinese Journal of Engineering Mathematics, 2007, 24(1): 87-94

多孔介质中两相可混溶驱动问题的特征-混合动态有限元方法

付红斐[1]， 芮洪兴[2]

(1- 中国石油大学数学和计算科学学院，东营 257061； 2- 山东大学数学学院，济南 250100)

摘 要: 数学上，多孔介质中一种不可压流体对另一不可压流体的相溶驱动由两个耦合的非线性偏微分方程组成，其中一个是关于压力的椭圆方程，另一个是关于浓度的抛物方程。本文用特征有限元方法结合动态有限元空间来逼近浓度，而压力和达西速度则由混合元方法来同时逼近。通过采用负模估计，我们给出了收敛性分析与误差估计。

关键词: 动态有限元空间； 修正的特征线方法； 混合元方法； 负模